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COMPLEXITY IN ECONOMICS: AN UP TO DATE VIEW

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Abstract:

It is well known that nonlinearity, economic dynamics, endogenity, and chaos, are some of the subjects closely related to complexity, and because of this all them must go on together towards a better knowledge of the behaviour of many systems in nature and society. It is for this reason that the progress and achievement as regard to the theory of complexity is reflected in related subjects, among which we emphasize the economic dynamics. Perhaps would be necessary a reconsideration of the idea or concept of complexity, especially if we are thinking in its relations with Economics. In this sense we try in this article to find a wider and open concept of complexity, taking into account that complex systems involves the integration of concepts from dynamics, information, computation, and evolution, and that as consequences of this integration would have to forge a new conceptual vocabulary and a new kind of mathematics. As an example we have chosen the analysis of capital market and time series, although there are many others works that have explored the potential applications of chaos theory to reveal “insights” into the structure and the dynamics of complex systems.

Keywords: Complex systems, complexity, economic dynamics, mathematics of chaos, computation, integration, capital market, time series.

LA COMPLEJIDAD EN LA ECONOMÍA: UNA VISIÓN ACTUALIZADA

Resumen:

Es bien sabido que la nolinealidad, la dinámica caótica, la endogeneidad y el caos constituyen algunas de las materias estrechamente relacionadas con la complejidad, y es por esta razón por lo que consideramos que todas ellas deben caminar conjuntamente hacia un mejor conocimiento del comportamiento de muchos sistemas en la naturaleza y en la sociedad. Recientemente se ha planteado la necesidad de una reconsideración de la idea o concepto de complejidad, especialmente en el campo de la Economía, y en este sentido tratamos en este artículo de encontrar una definición más amplia y abierta de la complejidad, teniendo en cuenta que los sistemas complejos implican o abarcan la integración de conceptos de dinámica, información, computación y evolución, y que como consecuencia de todo ello habría que forjar un nuevo tipo de vocabulario y una nueva clase de matemáticas. Como ejemplo de cuanto decimos en este trabajo, hemos elegido el mercado de capitales y las series temporales, aunque hay otros muchos casos en los que puede igualmente aplicarse la teoría del caos y su formalización en el marco de la estructura y la dinámica de los sistemas complejos.

Palabras clave: Complejidad, sistemas complejos, nolinealidad, caos, computación, integración. Mercado de capitales, series temporales.

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Introduction

The science of complexity has been used to refer the study of systems that operate at the edge of chaos, to infer structure in the complex properties of systems that are intermediate between perfect order and perfect disorder, or even as a simple restatement of the cliché that the behaviour of some systems as a whole can be more than the sum of their parts. The new concepts emerging from numerous research programmers, institutes and scientific journals specialized in this matter are since a certain time influencing disciplines as disparate as Astronomy, Biology, Physics and Economics, among others.

In the year 2001 the prestigious scientific review Nature said that the science of complexity was in its infancy, and although some research directions might eventually prove to be academic cul-de sacs, it seems reasonable to suppose that the general principles emerging from these studies will help us to better understand the complex world around us.[1] However years before, in 1988 concretely, a group of physicists and economists under the direction of two Nobel Prize, Philip W. Anderson and Kenneth J. Arrow, edited a set of articles where the Economy was considered as an evolving complex system. We are talking about the Proceedings of the Evolutionary Paths of the Global Economy Workshop held September 1987 in Santa Fe Institute and published one year later by Addison-Wesley Publishing Company.

At least in the field of Economics at present we think is more appropriate to use the expression complex economic dynamics, that in its broadest terms is concerned with the systematic study of changes and the forces generating it; that is, it is the attempt to determine how things change, why change occurs, and what kinds of change might come to pass in the future. We can mention some salient facts of economic change, as microeconomic and macroeconomic fluctuations, cycles, growth, development, explosive change and economic or financial crisis like the one we have been experiencing in recent years[2]. In many fields of science we can find outstanding advances in the recent years relative to the idea, development and application of complexity. For instance, this is the case of the Statistical Mechanics, whose concepts and methods have been infiltrated into many branches of science, engineering, and mathematics: ensembles, entropy, Monte Carlo, phases,

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[1] NATURE (2001): Vol. 410, nº. 6825, p. 241.
[2] FERNÁNDEZ DÍAZ, Andrés (2015): pp. 1-26.

fluctuations and correlations, abrupt phase transitions and criticality. All them are central to physics and chemistry, but also play key roles in the study of dynamical systems, communications, bioinformatics, and complexity. This type of books are careful to address the interests and background not only of physicists, but of researchers in mathematics, biology, engineering, computer science, and the social sciences, especially in economics.[3]

It is well known that nonlinearity, economic dynamics, endogenity, and chaos, are some of the most important matter closely related to complexity, and because of this all them must go on together in their walk towards a better knowledge of the behaviour of many systems in nature and society that are far too complex to analyze directly. It is for this reason that the progress and achievement as regard to the theory of complexity is reflected in related subjects, among which we emphasize the economic dynamics. Without stop we shall try to test in the following pages the degree of advance overtaken in the last fifteen years.

Complexity

The paradigm of complexity constitutes an obligatory reference to the scientific analysis on the verge of the XXI century, and this is especially true in the field of Economics, which is accustomed to navigate against the tide, a subject to a continuous dispersion, such as the one we had the occasion to emphasize in a paper we published some years ago.[4] The appeal of the complexity, in the case of Economics, warrants its interpretation as a healthy exercise, worthy of defence before the reductionist determinism professed by our science throughout its history, succumbing all too often to the aesthetic pleasure of simplicity. Should it not be clear, and with a view to taking a stand, it should be stated, with no undue delay, that determinism is pernicious: when we are distracted we deliberately skip the complexity.

We obviously do not seek the disqualification of determinism in radical support of decisive indeterminism, nor enter into the protracted and inconclusive polemics set forth all around it. What really and powerfully attracts the attention is the fact that a natural science, such as Physics, may have spectacularly advanced, which supposes its quantum revolution through basis on the Heisenberg´s uncertainty principle and the well known interpretation of Copenhagen due to Niels Bohr, in as much as, it reigns over a rigid determinism in the most complex sphere of the spirit,

[3] SETHNA, James P. (2008): pp. 1-3.
[4] FERNÁNDEZ DÍAZ, Andrés (2000): pp. 39-44.

of the culture and society, or, tantamount to the same, of a social science such as Economics.

With a view to entrenching the question momentarily, and without hazard of remission to other more specific and detailed works on the matter, determinism could be understood as an abstraction and simplification to make everyday complexity intelligible, considering, for its part, indeterminism as a consequence of our inability to explain complication, due to the fact that we do not avail of sufficient information. From this point of view, indeterminism would then be a clear consequence of complexity.[5]

Below the broad parasol of complexity, we find ourselves in a fascinating world of concepts, terms and instruments, bustling, intertwined, and opening new horizons in almost all fields of knowledge. Dynamics, non-linearity, irregularity, order and chaos are only some of them, behaving as parts of an indivisible whole. Chaos, habitually considered a subset of complexity, constitutes, without doubt, one of the key pieces of the process, and due to this, we can speak within our scientific field of *Chaotic Dynamic Economics*.

But, can Economics be set forth and understood in terms of complexity?. Evidently, it can and should be done, especially when dealing with an empirical science situated within the scope or group of *socials*, as we have already seen, and if it is taken into account that complexity is consubstantial and ubiquitous. If we remember the three types of complexity considered by Atlan[6], it could be easily proven that Economics exhibits or presents all of them, both quantitative as well as the essentially qualitative type. In effect, besides the probabilistic natural complexity, the most proper and direct, there is the algorithmic complexity, for instance, in the example of computer processed models of equilibrium, such as those carried out by Scarf in his well known work on matter. Likewise, one might undoubtedly speak about complexity in the appreciation of Economics, which, on the other hand, involves recognizing and admitting the existence of a subjective indeterminism in the sense already brought to hand.

In this sense, and as said Herbert Simon, Economics can be considered as a “hard” science, given that the complexity of the problems dealt with cannot simply be reduced to analytically solvable models or discomposed into separate sub-processes. Perhaps because of this in the last years the emerging interdisciplinary

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[5] FERNÁNDEZ DÍAZ, Andrés; ESCOT, Lorenzo; GRAU CARLES, Pilar (2012): pp. 2-4.

[6] ATLAN, Henry (1991): pp. 9-38.

"sciences of complexity" have provided new methods and tools for tackling these problems, from complex data analysis to sophisticated computer simulations, having applied recently all them to the field of Economics.[7]

Complexity and chaos

To all this it is necessary to add a type of strict definition of complexity, which is habitually used in Economics in the more recent works of specialists. In them, the term *complex* is used, to refer to those cases in which dynamic long-term behaviour is more complicated than a fixed point, a cycle limit, or a torus; or tantamount to the same, when chaotic behaviour is produced.

Important lines or sub-headings of the Economic Analysis and Economic Policy fall full and can be included in the scope of the Economics of Complexity. In these scopes, subjects referent to monetary dynamics or Keynesian dynamics are undertaken, problems of inflation and unemployment, the determinants of endogenous cycles, growth and distribution models, the development of exchange rates, the existence of chaos in capital markets, or the non-linearity and chaos in time series. In all of them, one tries to count on new approaches and methods that allow an analysis closer to the truth or reality, and to the intrinsic and inevitably dynamic nature of economic phenomena. We shall come back to that later.

Within the specific framework of financial markets there are many studies that test for non-linear dependence on daily stock indices, or searching for evidence of chaos in the future prices of commodities. It is also very important to search for the implications of non-linear dynamics for reasons of financial risk management; the chaotic behaviour in exchange-rate series, or nonlinearities and chaotic effects in option prices. Finally, non-linear modelling with neural networks offers a very interesting and efficient approach for studying the prediction of chaotic time series, and has been utilized successfully in different branches and problems of our science.[8]

It is necessary to remember that the major problem in time series research is the difficulty of distinguishing between deterministic chaos and a purely random process, taking into consideration that the most important characteristic of chaotic dynamical systems is their short-term predictability. Chaos is at the same time disorder and determinism. Chaos, in principle, due that is apparently disordered, make non predictable its evolution. But, on the other hand, being deterministic, and

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[7] SCHWEITZER, Frank (Edit.) (2002): Introduction.

[8] TRIPPI, Robert (1995): pp. 467-486.

governed by systems of non-linear equations, it should be possible to predict and control once you know the mathematical relationships of the variables that influence it. As said Henri Poincaré is much better to look farther without having certainty, that don't look anything at all. Because of this we must to undertake the analysis of the main concepts, techniques and mathematics of chaos, that is, of all the weapons we need to know and to deal with an irregular and complex reality, as already we have pointed out.

Before of going on, however, it is necessary to know that complexity and chaos are intimately related to the concept of emergence. What does not emergence means?: neither a trivial thesis about the fact that things change, nor a property or set of properties which apparition cannot be predicted. In no manner emergence must be interpreted as a limitation to scientific knowledge provoked by a certain class of indetermination. Emergence is not a unified and rigid totality invulnerable to the contradiction and falsification. Taking into account the evolution approach, we can think that emergent evolution may be interpreted as an incessant flow of creative novelty, which implicate a special conception of the whole and the parts, farther away the simple and lineal idea of an additive process.[9] In reality there is a process of emergence when the behaviour of the overall system cannot be obtained by summing the behaviours of its constituent parts. That is, the whole is indeed more than the sum of its parts.

The habitual definition of chaos, which hallmark is the sensitive dependence on initial conditions, implies that there is no information within chaos, and it has neither form nor structure. For us, chaos may be complex and appear to be non-deterministic, but hidden within it is a wealth of information. If in an emergent phenomenon there is also some hidden information, given that, as we have seen, the whole became something more than the sum of its components, seems clear, first, the closed relation between chaos and emergence, and secondly, the help that the last one can render to the predictability of chaos. We must remember this very important consideration in the concluding remarks of this work.

The characteristics of irregularity and non-linearity are, among others, derived from the complexity of economic behaviour, which oblige, as stated at the beginning, the utilization of concepts and new instruments especially conceived to face challenges, which today arise within Economics, and of course, in other fields of

[9] FERNÁNDEZ DÍAZ, Andrés (1999): pp. 139-143.

knowledge. Amongst them, the Theory of Catastrophes, and very especially, the Mathematics of Chaos stand out.

The majority of authors coincide in as much as the Theory of Catastrophes and the Mathematics of Chaos can be considered as two approaches to a general theory of dynamics of discontinuities. Both have in common as a base the idea of a splitting or halving of the equilibrium at critical points, just as the fact that functional relation are, with greater frequency, of the non-linear type. But they differ in as much as some discontinuities are set forth on a great scale, as the Theory of Catastrophes, and others on a small scale: the Theory of Mathematics of Chaos[10]. The Theory of Catastrophes is therefore a special case of the bifurcation theory accredited originally to Poincaré, which contemplates the world as essentially uniform and stable yet subject to sudden changes, the unexpected, or discontinuities on a grand scale which are produced in certain variables of state.

It is well known that the starting point of the Theory of Catastrophes can be found in the works of René Thom and Christopher Zeeman, at the end of the sixties and the beginning of the seventies in the XX century. On other occasions we have, and at certain length, taken to hand this new mathematical method, in order to describe the evolution of forms in nature, by hazarding even some economic applications, and concretely, the problem of stagflation[11]. We shall not go into this any further. Instead, we shall center our attention on Chaos and its measurement, which has greater relevancy for the purposes of our analysis.

It is often said that chaos is a ubiquitous phenomenon which is produced everywhere and can be observed in all fields of Science. Thus we find chaotic systems in the Hamiltonians, in the three bodies of celestial mechanics, in the physics of fluids, in lasers, in particle accelerators, in biological systems, in chemical reactions, and as we shall soon see, in no small part of the behavioral forms within the field of Economics.

Chaos can be located through the function of ***strange attractors***, by following bifurcation diagrams, or by analyzing the intricate profile of figures of fractal geometry. It should not be forgotten, in this respect, that, as Giambattista Vico said, chaos is *the raw material of natural things that, shapeless, is thirsty for form, and devours all*. We know that the essential geometrics of chaos consist of stretching

[10] BARKEKEY ROSER, J.Jr. (1991): pp. 2-3.
[11] FERNÁNDEZ DÍAZ, Andrés (1987): pp. 104-111.
CASTRIGIANO, Domenico P.L. and HAYES, Sandra A. (1993): pp. 39-62.

and bending, as pointed out by Stephen Smale in his topologic transformations. In effect, the irregularity of movement is produced by a mechanism which is broken down into two actions. On the one hand, the spatial phase is stretched, by separating trajectories, and then it doubles back onto itself. Of course, all that is in relation with complexity, especially in the case of fractals, that may be considered its geometry.[12]

The exponents of Lyapunov serve to explain the first part of this process, on giving a measurement on the exponential separation of two adjacent trajectories. We can study local instability of a discrete system $x_{n+1} = f(x_n)$ in the Lyaponov sense measuring how two adjacent points separate with the iterated application of the function, that is,

$$|f^n(\mathrm{x}_0 + \epsilon) - f^n(\mathrm{x}_0)| = \epsilon^{n\lambda(x_0)} \tag{1}$$

The limit

$$\lambda(x_0) = \lim_{n\to\infty} \lim_{\epsilon\to 0} \frac{1}{n} \log \left|\frac{f^n(x_0+\epsilon)-f^n(x_0)}{\epsilon}\right| \tag{2}$$

or also:

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \log \left|\frac{df^n(x_0)}{dx_0}\right| \tag{3}$$

are both (2) and (3) path expressions of the Lyapunov exponent. In a general n-dimensional system there will be *n* Lyapunov exponents, showing each of them the average rate of expansion or contraction of the phase space in each of its *n* direction under the action of the dynamical system.

These Lyapunov exponents can be used to distinguish simple dynamic attractors from complex dynamics attractors, that is, those that we could call existing traditional attractors until the Lorenz contribution in 1963, *fixed points*, *limit cycles* and *quasi-periodic torus,* from chaotic or strange attractors.

We can say that the dynamics inside fixed points, limit cycles or torus is a simple dynamics in the sense that two orbits that start arbitrary near each other, always

[12] For a rigorous and complete study on fractals see:
AGUIRRE, Jacobo; VIANA, Ricardo L.; SANJUÁN, Miguel A.F. (2009): pp. 337-359.

remain adjacent, thus providing a guaranteed predictability on a long term basis. However, inside the strange or chaotic attractor the dynamical system is complex in the sense that although bounded inside the attractor is also locally instable, with recurrent but aperiodic cycles, and with sensitive dependence to initial conditions making predictions difficult beyond the very short term. Therefore "simple dynamics" and "complex dynamics", can be detected by means of the Lyapunov exponent. More concretely, a positive Lyapunov exponent, i.e. local instability, is a necessary condition for the existence of chaos.

A wider and open concept of complexity

Perhaps would be necessary a reconsideration of the idea or concept of complexity, especially if we are thinking in its relations with Economics. In this sense we must to remember that the economic science in most of its lines of thought, especially in the neoclassical one, have taken advantage of the reductionist determinism, on one hand, and of the use of the *ceteris paribus* clause, on the other. About the first we have make some considerations to begin our analysis and definition of complexity; regarding the second, we shall do later some reflections.

In the search for the roots a problem, such is the inequality, it is necessary to adopt an interdisciplinary approach, relating economics with others fields of science. In this subject of inequality, for instance, would be very important and enriching to enter into the complex framework of the social psychology, using all the concepts, instruments and tools necessary to arrive to a clear and rigorous explanation for this disturbing economic and social problem. Indeed, this way of behaving suppose in our opinion to enlarge or extend the very meaning of complexity, wider and open to all kinds of phenomena without barrier and border.

At present the importance of the psychology in the economy has been accentuated in a remarkable way, very especially in the field of microeconomics, each time with more frequency treated from the point of view of the behaviour. A relatively innovating bridge between this psychological-social approach on inequality and the economic properly said constitutes what recently is known as *Identity Economics,* studied and presented by Akerlof and Kranton in 2010, although ten years before already was published in *The Quarterly Journal of Economics* an article with these same ideas. [13] In fact, in the beginning of this seminal work, the authors clearly

[13] AKERLOF, George A. and KRANTON, Rachel, E. (2010).
AKERLOF, George A. and KRANTON, Rachel E (2000): pp. 1-2.

affirm that …*This paper considers how identity, a person´s sense of self, affects economics outcomes. We incorporate the psychology and sociology of identity into an economic model of behavior.*

Other example that we can take into account is that one relative to the urban dynamics as a very important chapter of the applications of chaos and complexity to economic science, considering that change in development is often assumed to be proportional to existing size or scale as

$$\frac{dP_t}{dt} = \lambda P_t \tag{1}$$

where λ denotes a growth rate, expressing the continuous exponential growth by the relation

$$P_t = P_0 \cdot \exp(\lambda\, t) \tag{2}$$

This relation has been used in traditional models of cities to simulate the decline in densities, rents, trips to work, and so on, and also to model population growth in rapidly growing cities. These models are sensitive to their initial conditions in that their ultimate states are likely to be conditioned by their starting values. This sensitivity is characterized by what some author call *path dependence,* defined as different trajectories that emerge from the application of particular initial conditions. In fact, path-dependent behaviour as well as bifurcations is usually associated with more complex models, where the state is a function of many variables that refer to interacting places and activities.[14] The types of change affect the entire system, and that in physics is called a phase transition is also characteristic of urban systems qualitative and can be seen at many levels.

Recent works about urban dynamics outline ways in which cities grow to become more complex, moving from simple non-fractal forms to the point where their locational and relational structures of their activities and interactions are poised at equilibrium between order and chaos. With others words, this is the contemporary view of how fractals form, and it relates the morphology of cities to networks, phase transitions, self-organized criticality, and the "edge of chaos".[15] In certain measure, at present, urban structure and dynamics may be considered as "fractal geometry",

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See also: CALLERO, Peter L. (2014): pp. 273-293.

[14] BATTY, Michael (2007): pp.24-29.
[15] BATTY, Michael (2007): pp. 457-460.

which allows a more modern and rigorous analysis of emerging ideas in the economics of cities. In effect, as says Batty, in the process of generating theories for cities that are based on their essential dynamics, a new view of systems in general has arisen, and a new paradigm grappling with essence of such systems began to emerge supported by complexity theory and the physics of far-from equilibrium structures.

Processes that lead to surprising events, to emergent structures not directly obvious from the elements of their process but hidden within their mechanism, new forms of geometry associated with fractal patterns, and chaotic dynamics, all them are combining to provide theories that are applicable to highly complex systems such as cities. It is necessary to add that most of the developed models in urban dynamics are cell-based methods that can be categorized as models that use cellular automata, a technique widely applied in simulation of all kinds of spatial issues. In short, the future development of the economics of cities or urban dynamics find a valuable help in the principles and techniques of complexity and chaos, using new kinds of computational models and power laws, whose meaning, applications and interpretation constitutes a very exciting task.

Using large data sets Gopikrishnan et al (1999) and Mantegna and Stanley (1995) established a strong case for inverse “cubic” power law of stock market returns. They established that for several stock market indexes and U. S stocks the law is

$$P\left(|r_t| > x\right) \cong x^{-\alpha} \tag{3}$$

where α is in the vicinity of 3 or 4. This implies a kind of universal pre-asymptotic behaviour of financial data at certain frequencies. In the case of volatilities, measuring them as the absolute returns the power law is characterized by

$$Cov\left(|r_t r_{t-\Delta t}|\right) \approx \Delta t^{-\gamma} \tag{4}$$

where $\gamma \cong 0.3$ is a rather typical finding, which implies a strong correlation of volatility over time. This dependence can be used for portfolio and risk management. The measurement of long-range dependence is based on estimation of the Hurst exponent or the Detrended Fluctuation Analysis.

Also trading volume and number of traders exhibit power laws, with exponents similar in different types of markets and different countries. Gabaix et *al.* propose a model to explain these empirical power laws. Their model is based on the hypothesis that large movements in stock market activity arise from the trades of large participants, such as large fund managers, and power laws appears when trading behaviour is performed in an optimal way.[16]

This other example of the urban dynamics, considered and studied as a complex phenomenon or system, show clearly the necessity and advantages of opening the parenthesis and take out the ceteris paribus clause so habitually used in different schools in the field of the economy.

Applications to capital market and time series

Seem clear that that the present crisis is something more than the financial one inaugurated on January 2008, and it is so because there is a set of very important institutional, political and ethics factors that have contributed to flow situation into the present.

We must remember that when the stock market is studied, the behaviour of the set of chartist and fundamentalists is usually considered by utilizing equations such as the following:

$$P_{t+1} = P_t^{\alpha} \cdot F^{\beta(P_e - P_t)} \; ; \; \alpha > 0; \; \beta > 0 \qquad (1)$$

which would come out like this:

$$P_{t+1} = P_t^{\alpha} \qquad (2)$$

if the price of shares, P_t coincides with the balance price P_e, depending then the prices on the period t+1 solely of the chartist behaviour.

[16] GABAIX, X.; GOPIKRISHNAN, P.; PLEROU, V. and STANLEY, H.G. (2003): pp. 267-270.

By giving different values to the parameters α and β, and for a certain balance price, diverse results can be obtained that illustrate the evolution of the market. Let us suppose that P_e=2 and β=1, and that we take values of α at a certain interval $1< \alpha < 2{,}9$. For α=1,25 we obtain a fixed point, or a period cycle 2, if we vary the parameter and we make it equal to 2. If within the interval marked we assign to α the value 2,4, we would have a cycle period 4, leading to a chaotic situation if we raise it to 2,8.

In this way, we can obtain the corresponding Feigenbaum bifurcation diagram, in which the successive halving of the orbits can be seen, until the zone of chaos is reached, by appreciating some windows in their interior.

For the different values of α, and by maintaining β=1 and P_e=2, the existence of positive Lyapunov exponents corresponds to chaotic movement, as much as the negative values denote the presence of a regular or periodic evolution. On arrival to this point, it is necessary to remember that under the irregular and disordered behaviour, apparently random, it is possible to find a deterministic law and explanation, and as consequence of that, the capability of prediction.

Coming back now to the causes and evolution of the present crisis, would be easy to find, effectually, as we said before, that the ethics, political and institutional factors can show and explain the source of irregularities and disequilibrium in the international financial markets and economies, something that is well known by all operators, but it is necessary to demonstrate.

Very simply we can express the impact of these factors starting of the following functions:

$$I=f\,(c, L) \quad ; \quad P=h\,(g, t_r) \quad ; \quad E=\varphi\,(b, r, o)$$

where I, P and E mean, respectively, the institutional, political and ethics variables. In his stead, we must clarify the meaning of the independent variables corresponding to each of the functions that we express below:

c=corruption

L= rule of law

g= public management

t_r= public transparency

b=broker behaviour

r=rating firms behaviour

o= oligopoly power of rating firms

Then we could write the equation (2) in this other manner:

$$P_{t+1}=\Psi(\ I,\ (I,\ P,\ E) \qquad (3)$$

that is, expressing the price of shares on the period t+1 depending of observable variables that make possible to explain and predict the future behaviour in the capital market. With other words, we substitute (or complete) the chartist component, that is, the equation (2) by this last function in which the expected prices of the shares on t+1 depend upon the institutional, political and ethics independent variables.

From a methodological point of view we have behaved in inverse way, finding the deterministic law hidden in the chaotic dynamics referring to the evolution of the price of shares. If working with the equation (1) we can verify the existence of chaos with the help of the different tests (Lyapunov exponents, R/S, BDS, correlation dimension, spectral analysis, and so on), it means that there is a clear explanation of the financial phenomenon object of our analysis, although disguised of irregularity and disorder, that it is necessary to discover. In the case we have presented, the existence of deterministic chaos is also confirmed by the approach we have just exposed.

In a former work we have realized an empirical analysis of the capital market at Spain in the period 1941-1998. In this analysis were used two different series. The first was composed of the monthly data of the General Index of the Madrid Stock Exchange (GIMSE), since January 1941 until January 1998. The second was composed of the IBEX 35 daily data from January 1987 until March 1998, that is, a time series composed of 2776 data.

The study was based on the return stock-market data, and we found evidence of non-normality, positive Lyapunov exponent, evidence of non-linearity and, in short, signs of the existence of chaos, as can be verify with all kinds of detail in the pages dedicated to this application in our book concerning the chaotic dynamics in economics.[17]

In a very recent article we have extend the analysis to the years of the present crisis (2007-2013), taking into account our previous considerations, and we find that in this period it is highly probable the existence of chaos, principally when, in this case, we know the hidden law or deterministic explanation.

In the concrete analysis presented in this new article that obviously not going to play here, we have choose the evolution of the Madrid stock exchange during the lapse of time 2006-2013 that embraces the years of crisis that has its origin in the behaviour of financial markets.

This period is characterized by strong oscillations and high volatility in the markets that allows that techniques we know can throw light on the dynamical followed by the stock market in these hard times, specially starting from the collective stock exchange hysteria on Monday 21 January of 2008, due to the fear of one recession in the Unites States economy after the beginning of mortgage enormous fraud.

The analysis in this last mentioned work was realized also with the time series of IBEX 35 daily data in the course of the referred period, studying in such an analysis the time evolution of IBEX daily returns obtained as the composed returns or the first difference of logarithm of stock prices. The series that take daily data from January 2006 to December 2013 is integrated by 2019 data, and its time evolution appears in the figure nº 1 that we allow ourselves to reproduce here for further illustration. [18]

[17] FERNÁNDEZ DÍAZ, Andrés and GRAU CARLES, Pilar (2014): pp. 209-231.
[18] FERNÁNDEZ DÍAZ, Andrés (2015): pp. 19-31.

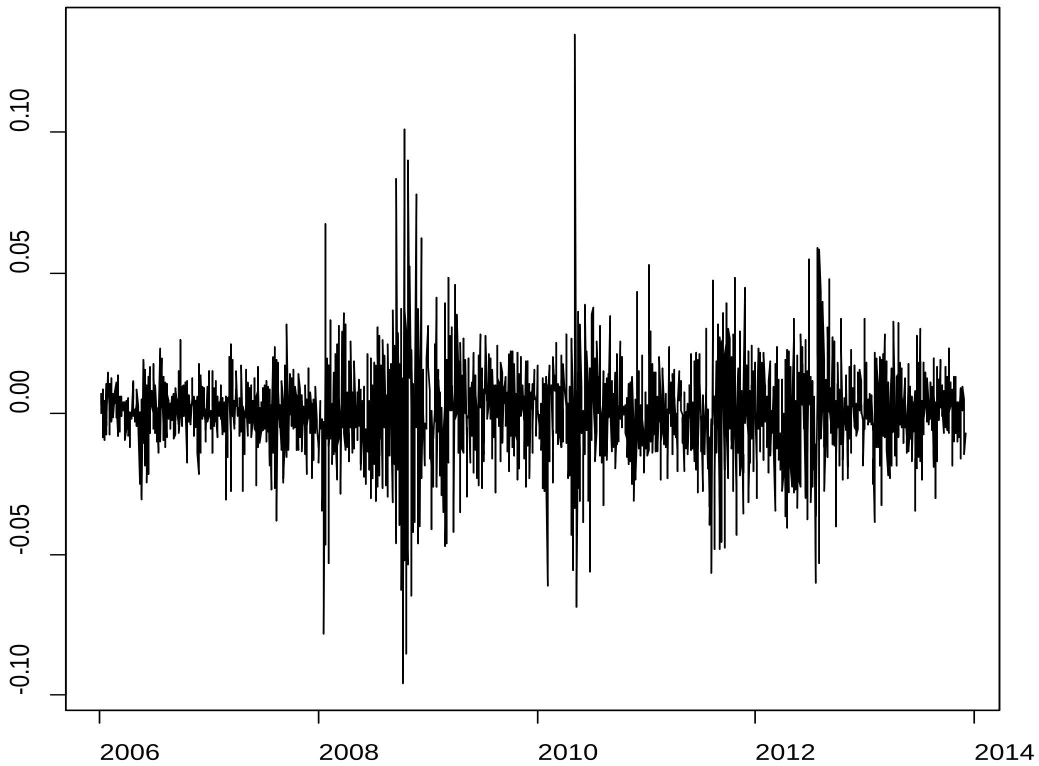

Figure 1: Time evolution of the IBEX 35 daily returns (2006-2013).

According to the statistical summary of data, the daily returns average for this period is very small, close to 0, while the daily volatility is around 1,7%. The smallest daily return is of -9,6%, whereas the largest is of 13,4%. The returns present a positive and small asymmetry, and a high kurtosis. These results are summarized in the Table 1. Likewise the hypothesis of normality of the data is rejected using the Jarque-Bera, and the presence of autocorrelation with the Ljung-Box test.

Table .1: Summary of statistical data

Max	13.484%
Min	-9.586%
Average	-0.005%
Standard Deviation	1.653%
Asymmetry	0.156
Kurtosis	8.823
Jarque-Bera (p-value)	0.000
Ljung-Box (20 retards (p-value)	0.000

We cannot repeat the analysis realized in our article just published with the title Chaos and fractal impact on economics, nor to reproduce the set of complicated graphics or figures that can be found in such the article already mentioned (see the previous footnote on page), but we can summarized the conclusions that were reached in our new application.

After using the various tests for detecting chaos and considering all results obtained in our empirical analysis, we cannot conclude clearly that the IBEX 35 daily returns in the period of crisis be covered by a deterministic system. However the series could also be described as a nonlinear dynamic system with noise far from the equilibrium, submissive to bifurcations or changes in the dynamical of the system. The problem with this approach is that although the dynamical of the system it could be modelling correctly, could not obtain better predictions.

Nevertheless it is necessary to emphasize that ethic, politic and institutional factors during the period analyzed can explain the irregularities and disequilibrium in the economy and in the international financial markets. The existence of corruption, the behaviour of the firms of rating with oligopoly power in the evaluation of the economic and financial policy of the different countries that work in the markets, and colluding with groups of brokers, are key elements in explaining the unusual and unconventional operation of capital markets in the years of the crisis. All that can be interpreted as evidences of that there has been something more that hazard, or what is the same, that there has been a chaotic behaviour in the evolution of the series of values, and has been seen, some ability to predict.

In the development of the Economic Theory, first, and in the application of the correspondent economic policy, later, may be observed that very often the independent variables that we have just considered, or others of similar nature and importance, are no taken into account, remaining jailed between the parenthesis than enclose the components of the clause *ceteris paribus.* Complexity in economics requires viewing reality as a whole without exclusions for reasons of analytical convenience.

In the study of the time series corresponding to the IBEX 35 in the years of the recent crisis, we have proceeded in such a manner, applying at the same time the diverse tests of chaos, arriving to a positive and interesting result in our analysis that can help in decision-making in capital markets.

It is important to stand out and to insist on the fact that with the application of these techniques of chaos, as we have seen in the example chosen, it is possible to distinguish between determinism or randomness when we are working with a time series. But we must point out that among statements that can be made relative to complexity there is one according to wihich the systems are very often neither completely deterministic nor completely random, and exhibit both characteristics. The answer or solution can be found, as we know, in the results when we apply the different tests to detect nonlinearities and the existence of chaos in a given time series.[19] In any case we are in conditions of obteining a better degree of information, and consequentely, a less degree of uncertainty about the phenomenum object of study.

When, as we have shown, it is found that a time series does not behave randomly due to the existence of chaos, ie, because underlies a causal relationship, it may be considered as a indication that the prediction is possible. All this despite the trend to identify chaos whith unpredictability, something that we reject, although it is necessary to recognized that reasonably accurate predictions would only be possible for very short time horizons.

[19] ÇAMBEL, Ali Bulent (1992): pp. 1-23.

Anything else?

Complexity is something more that a broad parasol of the non-linearity, irregularity, order and chaos, as we said at the beginning of these pages. Complexity, as science of complex systems, implies a challenge to integrate a large number of disciplines besides of the main and already mentioned. These disciplines are the statistical physics, stochastic process theory, the theory of information, the networks theory, the biology and the computational sciences. But we can go farther and refer to the Biophysics and the Astrobiology as two emerging sciences in which the complexity is very high. So much so that the biologic systems as part or component of Biophysics appears as paradigm of complexity and organization, something that we can understand easily if we analyze the relation of the diverse hierarchic levels of organization of the biologic systems with the distinct parts of physics.[20]

With respect to Astrobiology, that studies the origins of the Universe and the Life, we can affirm without doubt that is an emerging science with a huge degree of complexity. Effectively, as says the Spanish physicist Juan Pérez-Mercader, trying to understand if life is a cosmic imperative, that is, that it emerges everywhere in the Universe where there is an opportunity for chemical evolution, requires that we ask these questions within the framework of *complexity theory*, which becomes the basis analytical tool for describing, understanding and unifying astrobiological phenomena.[21]

Other scientific field of great interest in relation to complexity is de denominated *Synthetic Biology,* integrated by several scientific disciplines such as the nonlinear dynamics, the physics of the complex systems, engineering and the molecular biology. The goal of synthetic biology is to create artificial organisms, for which it is essential to understand what life is. Metabolism-replacement systems, or (*M-R*)-systems, constitute a theory of life developed by Robert Rosen, characterized in the statement that organisms are closed to efficient causation, which means that they must themselves produce all the catalysts they need.[22] In short, it is a new

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[20] MONTERO, Francisco y MORÁN, Federico (1992): pp. 15-29.

[21] PÉREZ-MERCADER, Juan (2001): pp. 337-358.
FERNÁNDEZ DÍAZ, Andrés (2013): pp. 5-8.

[22] CORNISH-BOWDEN, Athel; PIEDRAFITA, Gabriel; MORÁN, Federico; CÁRDENAS, María Luz; MONTERO, Francisco (2013): pp. 383-390.
ROSEN, Robert (1973): pp. 1-9.

emerging field of interdisciplinary nature in the sciences of complexity, from which significant progress is expected.

Among the many attempts to explain what is meant as complexity we have the book published by the physicist Peter Erdi with the title *Complexity Explained,* where basically explains the research in complex systems of nature and social phenomena and its importance in the general framework of science. In the book it is shown that very different complex phenomena of nature and society can be analyzed and understood by nonlinear dynamics from many systems of different fields, such as physics, chemistry, biology, economics, psychology and sociology, among others.[23]

Other outstanding book on complexity is that one wrote by Melanie Mitchell in 2009, in which purposes the following definition of the terms complexity system: *a system in which large networks of components with no central control and simple rules of operation give rise to complex collective behaviour, sophisticated information processing, and adaptation via learning of evolution.*[24]

In the chapter 19 of her book, dedicated to the past and future of the sciences of complexity, the author says that in her view complex systems science is branching off in two separate directions. Along one branch, ideas and tools of complexity research will be refined and applied in an increasingly wide variety of specific areas, many of them we are repeatedly mentioned, as physics, biology, epidemiology, economics, sociology, political science, and computer science, among others. The second branch would treat of considering all these fields from a higher level, so as to pursue explanatory and predictive mathematical theories that make commonalities among complex systems more rigorous, and can describe and predict emergent phenomena.[25] Mitchell adds that much work in complex systems involves the integration of concepts from dynamics, information, computation, and evolution, and that as consequences of this integration would have to forge a new conceptual vocabulary and a new kind of mathematics.

To conclude these reflections, and although we have already said something about it, is possible to sum up that complexity, as science of the complex systems, suppose a challenge to the integration of many matters, among which it is

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[23] ÉRDI, Péter (2008): pp. 7-20. 359-360.
SANJUÁN, Miguel A.F. (2009) : pp. 225-234.

[24] MITCHELL, Melanie (2009): p. 13.
DURLAUF, Steven N. (2012): pp. 50-51.

[25] MITCHELL, Melanie (2009): pp. 291.301.

necessary to point out the nonlinear dynamics and the theory of chaos, the statistical physics, the theory of stochastic process, the theory of information, the network theory, the biology and the computation sciences.[26]

Regard to Economics, we have seen that this science has been taken into account in all the approaches that we are considered and that are coming from the diverse fields of knowledge. One of the applications to our field from the point of view of complexity may be found precisely in the commented book of professor Péter Érdi, that in the chapters 3 and 4 deals with the nonlinear dynamic in economics, analyzing, as examples, the business cycles, as the Goodwin, Lotka-Volterra, and Kaldor-Kalecki models.

Again talking about the thought of economists, let us remember that in the year 2013 Brian Arthur wrote that complexity economics holds that the economy is not necessarily in equilibrium, that computation as well as mathematics is useful in economics, that increasing as well as diminishing returns may be present in an economic situation, and that the economy is not something given and existing but forms from a constantly developing set of institutions, arrangements, and technological innovations.[27]

All these affirmations about what complexity economics holds are accepted by the generality of economists, with the exception of some radical groups or schools, or with others words, of those who do not consider or respect the reality. Therefore, in principle, there is nothing new what it says Brian Arthur. However it helps to clarify the complex nature of our science and to recognize the need to limit the use of the terms *ceteris paribus,* with what that means.

The peculiar title of this last part of our article remember us a magistral passage of the renowned book *A Portrait of the Artist as a Young Man* published by James Joyce in 1916 (pp.155-157), but of course is nothing to do with the confession of our sins, given that our work at present deal with the difficult task of founding a more up-to-date and satistactory concept of complexity considered in general and related to the various fields of science, especially in that one of economics.

Of course the application of the techniques and mathematics of chaos to capital markets that we have chosen as an example is not the only possibility, because there are numerous excellent works that have explored the potential applications of chaos theory to reveal "insights" into the structure and the dynamics of complex

[26] LEÓN, Manuel de, and SANJUÁN, Miguel A.F. (2009): pp. 91-97.
[27] ARTHUR, W. Brian (2013): p.1.

systems. In this exploration have been studied the main subjects of Economics, already mentioned in other headline of this article, such as distribution, cycles and growth, general equilibrium, desequilibria, regulation of demand, inflation and unemployment, rate of exchange, and so on.[28]

And now a last consideration. Complexity, so present and implicated in all the areas of the knowlwdge, as we said before, need a new vocabulary and other tipe of mathematical formalization. As our main objetive in this article has to do with the field of economics, appear reasonable to expose an example that can enlighten us on the subject. It is the application of optimal control theory to making economic policy decisions.

We can take as point of departure the vector of variables of state or objectives, x, and the vector of the control or instruments variables, u. We would then:

$$\mathbf{x} = (x_1, x_2, \ldots\ldots x_n)$$

$$\mathbf{u} = (u_1, u_2, \ldots\ldots u_n)$$

where $x_1, x_2, \ldots\ldots x_n$ indicate the set of objectives variables , and $u_1, u_2, \ldots\ldots u_n$ that one of the instruments or tools.

Of course the variables of both vectors are not only those of character strictly economic, but also those relative to others fields connected with the economy, such as physics, biology, sociology, psychology, etc, as something characteristic of complexity.

The feedback process between the instruments or control variables and those of state or objectives, allows optimally economy evolve if complains with the condition of maximizing the functional

$$J[\mathbf{x}, \mathbf{u}] = \int_0^T \mathrm{F}[\mathbf{x}, \mathbf{u}, t]\mathrm{dt}$$

[28] FERNÁNDEZ DÍAZ, Andrés (1994): pp.13-136.
FERNÁNDEZ DÍAZ, Andrés; GRAU-CARLES; Pilar and ESCOT, Lorenzo (2002): pp.469-482.

the evolution of the system being defined by the state equations:

$$\dot{x}_i = \frac{dx_i}{dt} = f_i[\mathbf{x}, \mathbf{u}, t] \qquad (i = 1, n)$$

The control variables, can be considered observables, and only can take values within a restricted set Ω. Besides hold a qualitative component that is necessary to consider in the measurement in order to get to quantify its impact on the objectives variables. But we are treating with a cardinal approach, and as such one there must be procedures and mathematical instruments to find a quantitative solution. In fact, to solve the problems derived of this approach have been habitually used three classic techniques: the calculus of variations, the principle of Pontryagin, and dynamic programming.[29] In the matter, we stand out the fact that the concept of *functional*, and most precisely, the *Functional Analysis* constitutes a powerful mathematical tool that has not been enough profited in the field of Economics.

In a very easy and simple way we can define a *functional* as a linear mapping of a functional space in other one. In his stead, functional analysis is the study of topological vector spaces which elements are functions. This matter discusses modern theories of differentiation and integration and the principal problems and methods of handling integral equations and linear functional and transformations.[30]

Disciplines as the optimatization theory, calculus of probabilities, stochastic process, the theory of differential equations and partial derivatives, etc, play a vital role in the study of capital markets and, in general, of the financial economy. Also we can affirm that the functional analysis, as already we have seen and the measure theory may be considered important tools to know the behaviour and results of the financial systems, increasingly complex and in need of a powerful new mathematical formalization. With respect to this matter, we find a fruitful source of mathematical foundations in the large number of papers published by Henry Poincaré on the so-called automorphic and fuchsian functions, on differential equations, and on topology, treating with great mastery of technique and full understanding all pertinent fields of pure and applied mathematics.

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29 See: FERNÁNDEZ DÍAZ, Andrés (1976):. pp. 169-183

30 For a deep study of this subject see:
RIESZ, Frigyes and Sz-NAGY, Béla (1990): pp. 341-360.

Let us remember that automorphic functions are the generalization of the circular, hyperbolic, elliptic, and certain other functions of elementary analysis. We know that a circular function, as sin z, has the property that it is unchanged in value if z is replaced by z + 2mπ, where m is any integer; that is, the function is unaltered in value if z be subjected to a transformation of the group z´= z + 2mπ.[31]

It is interesting to point out that in relation with the need of working with the necessary mathematical instrument there is a known *Journal of Complexity* (Elsevier) that has a wide variety of areas as, among others, the following: Computational finance, computational stochastic, control theory, information-based complexity, Markov chain Monte Carlo, nonlinear and algebraic equations, numerical analysis, optimization, and Turing machine model.

We must add, following with our main subject, that models of complex systems can capture much useful information but can be difficult to apply to real-world decision-making, given that the type of information they contain is often inconsistent with that required for traditional analysis. New approaches, which use inductive reasoning over large ensembles of computational experiments, now make possible systematic comparison of alternative policy options using such models of complex systems.[32]

And now, to finish, it is necessary do not forget that complexity, as chaos, is ubiquitous, a property that requires us to be always up to date and progress in its analysis and research, because the general principles emerging from these studies will help us to be better understand the complex world around us, a universe that, as Karl Popper[33] has written, *is partly causal, partly probabilistic, and partly open: it is emergent,* and we should add…partly chaotic and inevitably complex.

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[31] STRUIK, Dirk J. (1948): pp. 278-281.
FORD, Lester R. (2004): pp. 83-85.

[32] LEMPERT, Robert J (2002): pp. 1-5.

[33] POPPER, Karl (1982): p. 130.

APPENDIX A: METRIC SPACES AND TOPOLOGY

For any real numbers x and y, the Euclidean distance between them is the number

$$d_E(x, y) = |x - y| \in \mathbb{R}^+$$

Generalizing this idea, a set having a distance measure or metric, defined for each pair of elements is called a *metric space*.

Let S denote such a set. Then a metric is a mapping d: S x S $\mapsto$ $\mathbb{R}^+$ having the properties:

1. $d(y, x) = d(x, y)$
2. $d(x, y) = 0$ iff if $x = y$
3. $d(x, y) + d(y, z) \geq d(x, z)$ (triangle inequality).

It is possible now define a metric space (S, *d*) as a set S paired with metric *d*, such that the precedent conditions hold for each pair of elements.[34]A metric space (S, *d*) it said is complete if every Cauchy sequence converges at an element x, that is, there is an element of the space that is the limit of the sequence.

Is advisable to clarify that the terms mapping, transformation, etc, are synonyms for function, but an extra usage is *functional*, which refers to the case where the domain is a space whose elements are themselves functions, with $\mathbb{R}$ as co-domain. An example may be found in the text of this article when we deal with the application of optimal control theory to economic policy, in which we introduce the condition of maximize a functional for getting the optimal evolution of economy.

Finally, we must point out that metric spaces form a subclass of a larger class of mathematical objects called topological spaces. These do not have a distance defined upon them, but the concept of open set, neighborhood, and continuous mapping are still well defined. Even though only metric spaces are encountered in the sequel, much of the reasoning is essentially topological in character.

[34] For a more developed analysis see:
FAISANT, Alain (1977): pp. 157-182.

Poincaré published a large number of papers on the so-called automorphic and fuchsia functions, on differential equations and topology. In this last branch his work was really outstanding, what is important taking into account its relation, subsequently, with chaos and complexity.

NOTE: Something about Topology.

The base of the classic concept of movement rested in the distinction between the underlying metric vessel and their variable physical content, that is, between empty and full. The indifferent geometric vessel does not exist in the general relativity theory. Thus the material is converted to local deformation of space-time medium. More precisely, what was considered as a material body is nothing more than a center of this deformation whose movement is accompanied by the concomitant movement of the whole.

This physical concept of deformation it is the underlying concept of topology. Indeed, and as discussed below, the deformation is a fundamental property of topological bodies. We can star from the concept of "structure" and their four types: algebraic, of order, topological and complex. The last one, that is, the topological structure, refers to neighborly relations, limit and continuity.

Topology or "Analysis Situs", that so is also denominating, is an extension within the field of geometry. Beyond the metric geometry and the projective one, inspired in the notions of distance and straight line, respectively, appears a third geometry, the "qualitative", in which don´t works the quantity: we are speaking of the Topology. In it two figures are equivalent always that been possible to pass of one to another due to a continuous deformation.

Let us suppose that we have a rubber ball. If we go by the pressure deforming without breaking or altering their nature, will have properties of primitive figure that will not change: these are the topological properties. Its development has allows the treatment and systematization of issues as the theory of dimension, the differential geometry of curves and surfaces, the Poincaré´s polygon, etc.

To conclude, it is relevant to point out that since many years the topology has been applied to Economics to solve important and complex problems, mainly in the specific field of the General Equilibrium, without forget others areas of our science.

APPENDIX B: MARTINGALES

We denote by

$$H_{t-1}= h\ (X_{t-1}, X_{t-2}, \ldots\ldots\ldots) \qquad (1)$$

an information set content in a stochastic process until the time t-1.

Then we can say that a stochastic process is a martingale if comply the following conditions:

1. There is a μ (t)⟶ ∀t, where

$$\mu\ (t) = E\ [\ X(t)\ ] = \int_{-\infty}^{\infty} x\ dF(x,t) \qquad (2)$$

2. Is verified that

$$\mu(\ t\ ) = E\ [X\ (t\ )\ /\ H_{t-1}\ ] = X(t-1) \longrightarrow \forall t \qquad (3)$$

what implies for any $\tau \geq 0$

$$E\ [\ X\ (t+\tau)\ /\ H_{t-1}] = X\ (t-1) \qquad (4)$$

We can now define the following concepts:

$$\text{Submartingale} \ldots\ldots\ldots\ldots E[\ X(t) / H_{t-1}] \geq X(t-1)$$

$$\text{Supermartingale} \ldots\ldots\ldots\ldots E[(X(t) / H_{t-1}] \leq X(t-1) \qquad (5)$$

$$\text{Semimartingale} \ldots\ldots\ldots\ldots E[\ X(t) / H_{t-1}] > X(t-1)$$

that is, the best predictor of a stochastic process starting from t is always the preceding observation X(t -1).

It is important to remember that is possible to express or study martingale in relation to random walk that, as we know, is one of the simplest random processes. We see without stop this possibility.

Let us begin with the definition of random walk:

$$Y_t = Y_{t-1} + \varepsilon \qquad (6)$$

where ε is a random variable which zero mean and a constant variance, and where there is zero correlation between observations.

With drift element that actually refer to a time trend the equation would be

$$Y_t = Y_{t-1} + \alpha + \varepsilon \qquad (7)$$

and depending of the value of α we should have the following:

If $\alpha = 0$, we have a random walk.

If $\alpha > 0$, we have a submartingale. (8)

If $\alpha < 0$, we have a supramartingale.

Martingale is a more general stochastic process than a random because need not have constant variance. The term drift is used to represent the positive or negative trend in the time series of the stochastic variable.

Finally, is important to emphasize that a martingale is always defined with respect to some information set, and with respect to some probability measure. If we change the information content and/or the probabilities associated with the process, the process under consideration may cease to be a martingale. The opposite is also true. Effectively, given a process X_t, which does not behave like a martingale, we may be able to modify the relevant probability P and convert X_t into a martingale.[35]

Note: White noise.

A time series is said to be "white noise" if the underlying variable has zero mean, a constant variance and zero correlation between successive observations.

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[35] To complete this exposition we recommend :
ROGER, Patrick (1991): pp. 187-200.

APPENDIX C: WIENER PROCESS AND BROWNIAN MOTION

A **Wiener process** W_t relative to a family of information sets $\{I_t\}$, is a stochastic process such that

1. The pair I_t, W_t is a square integrable martingale with $W_0 = 0$ and

$$E\,[(W_t - W_s)^2] = t - s; \qquad s \leq t \tag{1}$$

2. The trajectories of W_t are continuous over t.

Starting from this definition we have the following properties of a Wiener process:

1ª.-W_t has uncorrelated increments because it is a martingale, and because every martingale has unpredictable increments.

2ª.-W_t has zero mean because it starts at zero, and the mean of every increment equals zero.

3ª.-W_t has variance t.

4ª.-The process is continuous in the sense that in infinitesimal intervals, the movements of W_t are infinitesimal.

We must know that the Wiener process is the natural model for an asset price that has unpredictable increments but nevertheless moves over time continuously.

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Now we are going to give the definition of **Brownian motion**:

A random process B_t, $t \in [0, T]$ is a Brownian motion if:

1º.-The process begins at zero, $B_0 = 0$.

2º.-B has stationary, independent increments.

3º.-The B_t is continuous in t.

4º.-The increments $B_t - B_s$, have a normal distribution with mean zero and variance $|t - s|$:

$$(B_t - B_s) \sim N(0, |t - s|). \tag{2}$$

This definition is, in many ways, similar to that of the Wiener process. Then we can establish as a theorem that any Wiener process W_t relative to a family I_t is a Brownian motion process.[36] However there is an important difference consisting in the fact that W_t was assumed to be a martingale, while no such statement is made about B_t, that is posited has a normal distribution. In any case, this theorem is very explicit, given that we can use the terms Wiener process and Brownian motion interchangeably.

The Brownian motion constitutes one of the theories that have had more success in the study of the capital markets, and in its behaviour we find a clear parallelism between the displacement of a particle and the time series of assets returns, as the later seems to evolve without a fixed pattern and seemingly random. In this regard let us not forget that in 1923 Wiener introduced the concept of random Brownian motion function.

If we consider a normalized Gaussian random process { ξ }, the increase in the position of a Brownian particle is:

$$X(t) - X(0) \longrightarrow \xi \, | t - t_0 |^{H}, \text{ where } H=0{,}5 \qquad (3)$$

that is, if we want to know the position of a particle in a given moment t, knowing the position in a reference point t_0, a random number ξ that follow a Gaussian distribution is chosen, multiplied by $|t - t_0|^{0,5}$, and this result is added to the position in t_0, that is $X(t_0)$.

A generalization of this function is the fractional Brownian motion introduced by Mandrelbrot and Van Ness (1968). In the particular case of H=0,5 the random function corresponds to the ordinary Brownian motion.

[36] For a more deep analysis see:
NEFTCI, Salih N. (1996): pp. 143-167.

APENDIX D: DIFFUSION AND TELEGRAPH EQUATIONS[37]

The prototype diffusion equation, also called the heat equation, can be expressed in this way:

$$\frac{\partial \psi}{\partial t} = D\nabla^2 \psi \qquad (1)$$

that is,

$$\frac{\partial \psi}{\partial t} = D\left[\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}\right] \qquad (2)$$

where *D* is the diffusion coefficient.

In a one dimension it would

$$\frac{\partial \psi}{\partial t} = D\frac{\partial^2 \psi}{\partial x^2} \qquad (3)$$

whose solution would be

$$\psi(x,t) = \frac{1}{2(\pi D t)^{1/2}} e^{-x^2/4Dt} \;\; \rightarrow \;\; t > 0 \qquad (4)$$

Let us take a more simplify version of (4), where the coefficient D is implicit[38]

$$\psi(x,t) = t^{-1/2} \cdot e^{-x^2/2t} \qquad (5)$$

[37] There is a more extensive version of this last Appendix in our book:
FERNÁNDEZ DÍAZ, Andrés (2000): pp. 237-250.

For different values of *t*, and varying x between –6 y 6, could be obtained the graphic representation of the diffusion process, as we can see in the figure 1:

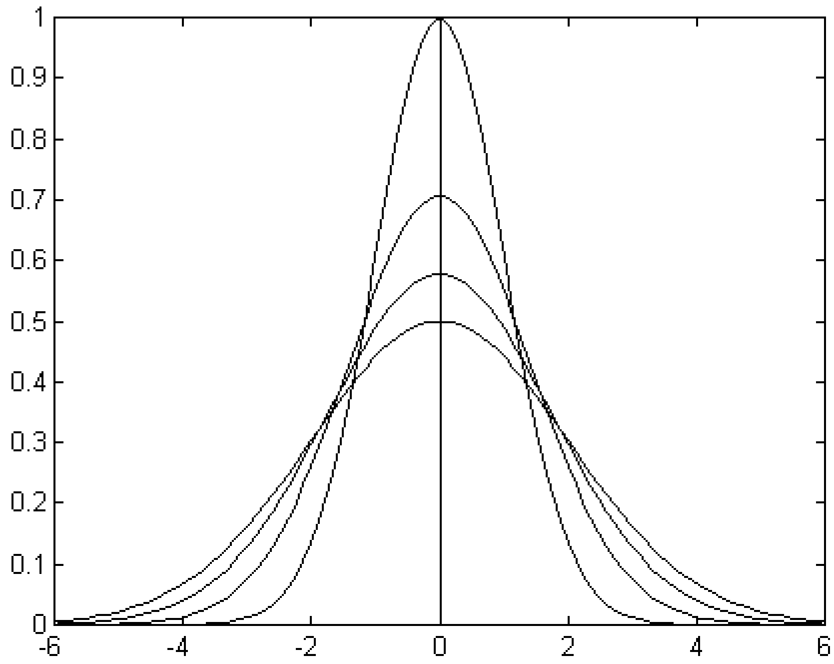

Figure 1

Note that the curve is having fewer slopes as it increases the value of t, remaining constant the surface regardless of that undergoes a process of diffusion. Of course, the analysis could be extended to the case of two and three dimensions.

Related to the diffusion equation we can to consider the so-call "telegraph equation", which takes the form

$$\frac{1}{a}\frac{\partial^2\psi}{\partial t^2}+\frac{\lambda}{a}\frac{\partial\psi}{\partial t}=D\frac{\partial^2\psi}{\partial x^2}+\beta\frac{\partial\psi}{\partial x} \qquad (6)$$

where D and β are adequately defined.[39]. The equation shows that an initial impulse propagates as a wave, and subsequently by a diffusion process. So, the sequence would be impulse ⇒ wave ⇒ diffusion ⇒ impulse...

[39] Note that in (6) are included the components of the wave equation, being this one, as we

Both the diffusion equation as Telegraph has a structure of great interest to explain some economic phenomena. In effect, this type of equations lead to power laws and to behaviour and scenarios of avalanches and self-organized critically that may be illustrative to study outstanding and complex fields of Economics.

As an example we will consider an interpretation of Black-Scholes model to market options by studying certain stochastic processes.

Let us start of S, the price of an asset that we can consider as a random variable that follows a Wiener process, expressing in the following way a change in a little interval of time:

$$dS = \varepsilon\sqrt{dt} \qquad (1)$$

where ε is a standard normal variable, being therefore *dS* normally distributed, with mean zero and standard deviation $\sqrt{dt}$.

Given that the different securities or bonds have volatility, multiply the second member of the equation (1) by σ, that is the annualized standard deviation of *dS*. It would then have:

$$dS = \sigma\varepsilon\sqrt{dt} \qquad (2)$$

But as it makes no sense a null change of price of asset, it would be necessary to add a parameter to represent the positive or negative trend in the time series of the stochastic variable (drift parameter).

The (2) would remain so:

$$dS = \alpha dt + \sigma\,\varepsilon\,\sqrt{dt} \qquad (3)$$

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know, and for only one dimension, $\frac{\partial^2\psi}{\partial t^2} = a^2\frac{\partial^2\psi}{\partial x^2}$.
For a mathematical explanation of Telegraph equation see:
BOYCE, William E.; DI PRIMA, Richard C. (1990): pp. 613-619.
See also MONTROLL, Elliot W. and WEST, Bruce J. (1992): pp.110-11.

indicating with αdt the mean or the expected value of the rate of return.

A generalization of a Wiener process is constituted by the Ito process, in which so the expected return (α) as (σ) are dependent on the underlying variables *S* and *t.*

The equation that would express the Ito process would be

$$dS = \alpha(S,t) + \sigma(S,t)\ \varepsilon\sqrt{dt} \qquad (4)$$

and more concretely

$$dS = \alpha S dt + \sigma S \varepsilon\sqrt{dt} \qquad (5)$$

With an Ito process we could value derivatives as forwards, futures and options, taking into account that any variable that is function of others that follow an Ito process also follows such one[40].

In this manner, if *W* is value of a financial product derived of other whose price is S, we can write:

$$dW(S,t) = \left[\frac{\partial W}{\partial S} aS + \frac{\partial W}{\partial t} + \frac{1}{2}\frac{\partial^2 W}{\partial S^2}\sigma^2 S^2\right] dt + \frac{\partial W}{\partial S}\sigma S \varepsilon\sqrt{dt} \qquad (6)$$

reflecting the formulation content in the bracket what we have denominate “drift rate”.

An expression of this type, well known and used in the financial analysis, is the Black-Scholes equation[41]

[40] See WATSHAM, Terry .J. & PARRAMORE, Keith (1977): pp. 334-343.
See also ITO, Kiyosi (1951): pp. 1-51.
[41] See: BLACK, Fischer and SCHOLES, Myron (1973): pp. 637-659.

$$\frac{\partial W}{\partial t} + rS\frac{\partial W}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 W}{\partial S^2} = rW \qquad (7)$$

where r indicates, in certain per one, the rate of official basic interest, or without risk.

Removing, the (7) would remain so:

$$\frac{\partial W}{\partial t} = rW - rS\frac{\partial W}{\partial S} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 W}{\partial S^2} \qquad (8)$$

Note that in the left side of the equation remains, in the limit, what in the terminology of the financial markets is called the Theta of a portfolio of options or of an option, Θ, that is the rate of variation of the value of the portfolio or of the very option over time.

Likewise, and following with the habitual terminology, in the right side we find what is known as the Delta of an option, Δ, which it is defined as the rate of change of it price with respect to the price of the asset underlying, or what is the same, the slope of the curve that relate the change of the price of the option *(ΔW)* with that one that feels the price of the stock that serves as base *(ΔS)*; in the limit we should have then the expression $\frac{\partial W}{\partial S}$, or the rate of the change $\frac{\partial^2 W}{\partial S^2}$.

The equation (8) would be equivalent to the following:

$$\Theta = \mathrm{rW} - \mathrm{rS}\Delta - \frac{1}{2}\sigma^2 S^2 \frac{\partial(\Delta)}{\partial S} \qquad (9)$$

Given, on the other hand, that the rate of change of the Delta of the option with respect to the price of the underlying asset may be defined, in his turn, as the Gamma (Γ) of a portfolio of options, or of an option over an underlying asset, we can arrive, finally, to the formula

$$\Theta = \mathrm{rW} - \mathrm{rS}\Delta - \frac{1}{2}\sigma^2 S^2 \Gamma \qquad (9)$$

However, and in the framework of our analysis, it is preferable to return to the formalization of the (8), thus avoiding any confusion in handling the Δ in the financial sense, on the one hand, and the laplacian of a dimension that is part of the equation, explicit or implicitly, on the other.

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